ECS 20 – Fall 2021 – Phillip Rogaway — Logic II (leading to **set theory**)

Adding quantifiers -- First order logic

"All apples are bad"

$(\forall x)\ (A(x) \rightarrow B(x))$ // universe of discourse?

"Some apples are bad"

$(\exists x)\ (A(x) \wedge B(x))$ // universe of discourse?

"BILLY has beat up every boy at the Caesar-Chavez elementary school"

$(\forall x)\ ((\text{Student}(x) \wedge \text{Boy}(x) \wedge (x \neq \text{BILLY}) \rightarrow \text{HasBeatenUp}(\text{BILLY}, x))$

// universe of discourse?

Universe of discourse = what quantifiers range over. Always important to know the universe of discourse; it's implicit or, better!, explicit in any discussion of logical formulas involving quantifiers.

All lions are fierce	$(\forall x)\ (L(x) \rightarrow F(x))$
Some lions do not drink coffee	$(\exists\ x)\ (L(x) \wedge \neg\ C(x))$
Some fierce creatures do not drink coffee	$(\exists x)\ (F(x) \wedge \neg\ C(x))$

"Nobody likes a sore loser"

Universe of discourse = human beings (is this really unambiguous?)
$L(x, y)$ - predicate - true iff person x likes y (is this really unambiguous?)
$S(x)$ - person x is a sore loser

$(\forall x)\ (S(x) \rightarrow (\forall y) \neg\ L(y, x))$

(apparently, a sore loser doesn't like even himself)

If anyone in the family gets COVID, then everyone
In the family must be quarantined.

universe of discourse - people
$F(x,y)$ – x and y are different people in the the same family
$Q(x)$ - person x must be quarantined
$C(x)$ - person x has the COVID

$$(\forall x)\ (\forall y)\ \ (F(x,y) \wedge C(x) \rightarrow Q(x) \wedge Q(y))$$

Discuss the mismatch / absurdity of trying to translate English into logical formulas

People like to speak of the variables as corresponding to declarative claims in English, either true or false, and they like to speak of our WFFs as modelling English-language sentences built around if, or, and, not. If it disingenuous. We don't use language in similar ways in math and in every-day language.

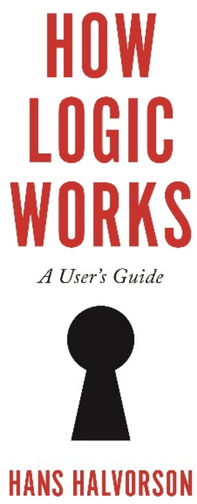
- For lunch, do you want Indian or Thai?
- **If** NSA computers store and analyzes everything you say on the phone or do on the internet, **then** democracy is over.

The first sentence is not to be answered yes, unless you are trying to be cute. The second sentence is expressing a causal or foundational matter; it cannot be replaced by

- If □ is irrational then democracy is over.

and preserve its meaning.
Situation doesn't get better when we add quantifiers. Don't take seriously any claim of a meaningful relationship between logic and natural language communications. And maybe only a weak connection to clear thinking. When someone say you are not being logical, we usually are failing to factor in the impact that having a different world view places on our interpretation of events. That we require different levels of evidence, or come in with different levels of skepticism. Or that we have different cognitive biases. Logic per se is rarely at the heart of it.

There are other points of view. Catalog that just came from Princeton University Press Philosophy …



How Logic Works shows that formal logic—far from being only for mathematicians or a diversion from the really deep questions of philosophy and human life—is the best account we have of what it means to be rational. By teaching logic in a way that makes students aware of how they already use it, the book will help them to become even better thinkers.

Formalizing First-Order Logic

Below, not a formal treatment, but a formal treatment can be found in any standard logic book, eg., Enderton. We can extend our treatment of propositional logic (sentential calculus) to first-order logic. But we won't do this formally:

Vocabulary of first-order logic consists of:

1. Parenthesis and logical connectives: ( ) $\neg \wedge \vee \rightarrow \leftrightarrow$
2. Equality symbol: = *usually included*
3. $\forall$, $\exists$ *universal and existential quantifiers*
4. Variables: $x_1, x_2, \ldots$ *name points in the universe U*
5. Constant symbols 0, 1, BILLY *name a point in the universe U*
6. Function symbols *f*, S,+, … *map a tuple of points in U to a point in U*
7. Predicate symbols P, Q, Prime, … *functions from universe U to* $\mathbb{B}$

Negating Quantified Boolean Expressions

PUSHING QUANTIFIERS

$\neg(\forall x\ \phi) \equiv (\exists x)(\neg\phi)$

$\neg(\exists x\ \phi) \equiv (\forall x)(\neg\phi)$

negate this:

$(\exists x)(\forall y)(y>x \rightarrow \exists z\,(z^2 + 5z = y))$

$\neg (\exists x)(\forall y)(y>x \rightarrow \exists z\,(z^2 + 5z = y))$
$(\forall x) \neg (\forall y)(y>x \rightarrow \exists z\,(z^2 + 5z = y))$
$(\forall x)(\exists y) \neg (y>x \rightarrow \exists z\,(z^2 + 5z = y))$

$\neg (A \rightarrow B) \equiv \neg (\neg A \vee B) \equiv (A \wedge \neg B)$

$(\forall x)(\exists y)(y>x \wedge \neg\exists z\,(z^2 + 5z = y))$
$(\forall x)(\exists y)(y>x \wedge \forall z \neg (z^2 + 5z = y))$
$(\forall x)(\exists y)(y>x \wedge \forall z (z^2 + 5z \neq y))$

Example: **negligible functions**
A function $f\colon \mathbb{N} \rightarrow \mathbb{R}$ is negligible if it vanishes faster than the inverse of any polynomial:

$(\forall c>0)(\exists N)(\forall n \geq N)\, f(n) \leq n^{-c}$ shorthand for
$(\forall c)(\exists N)(\forall n)(c>0 \wedge n \geq N \rightarrow f(n) \leq n^{-c})$

eventually, you're less than n^{-c} for ANY c. Negate it:

there is a c s.t., infinitely often, you're bigger than n^{-c}

Even grad students and researchers get confused about this!

$\neg (\forall c)(\exists N)(\forall n)(c>0 \wedge n \geq N \rightarrow f(n) \leq n^{-c})$
$= (\exists c) \neg (\exists N)(\forall n)(c>0 \wedge n \geq N \rightarrow f(n) \leq n^{-c})$
$= (\exists c)(\forall N) \neg (\forall n)(c>0 \wedge n \geq N \rightarrow f(n) \leq n^{-c})$
$= (\exists c)(\forall N)(\exists n) \neg (c>0 \wedge n \geq N \rightarrow f(n) \leq n^{-c})$
$= (\exists c)(\forall N)(\exists n)(c>0 \wedge n \geq N \wedge f(n) > n^{-c})$

Infinitely often, you are bigger than n^{-c}

Important Examples

Set Theory

predicate symbols: 2-ary $\in$
function symbol: $\varnothing$

Note "syntactic sugar" -- write $a \in A$ instead of $\in(a,A)$.
But that doesn't change that $\in$ is a 2-ary predicate.

"For any pair of sets, x and y, there a set $x \cup y$ that contains all of the elements of x and y"

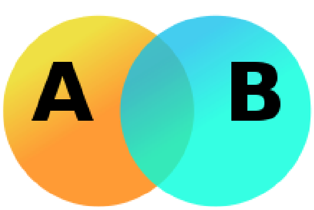
$$(\forall x)(\forall y)(\exists z)(\forall u)\ (u\in z \leftrightarrow (u\in x) \vee (u \in y))$$

Seems **very** spare, with just $\in$.
What are other operators on sets, and how would we define them?

$A \subseteq B$: (another 2-ary predicate)

$$a \notin A \quad = \quad \neg(a \in A)$$
$$A \subseteq B \quad ::= \quad (\forall x)(x\in A \rightarrow x\in B)$$
$$A \supseteq B \quad ::= \quad (\forall x)(x\in B \rightarrow x\in A)$$

Define
union ($\cup$)
intersection ($\cap$)
complement (A^c or $\bar{A}$),
symmetric difference $\oplus$
set difference ($A \setminus B$ or $A - B$)
formally, and illustrating with Venn Diagrams



Algebra of sets

$A \cup A = A$ $\quad$ $A \cap A = A$

$A \cup (B \cup C) = (A \cup B) \cup C$ $\quad$ $A \cap (B \cap C) = (A \cap B) \cap C$

$A \cup B = B \cup A$ $\quad$ $A \cap B = B \cap A$

$A \cup (B \cap C) = (A \cup C) \cap (B \cup C)$ $\quad$ $A \cap (B \cup C) = A \cap B \cup A \cap C$

$A \cup \varnothing = A$ $\quad$ $A \cap \varnothing = \varnothing$

$A \cup U = U$ $\quad$ $A \cap U = A$

$(A^c)^c = A$

$A \cup A^c = U$ $\quad$ $A \cap A^c = \varnothing$

$U^c = \varnothing$ $\quad$ $\varnothing^c = U$

$(A \cup B)^c = A^c \cap B^c$ $\quad$ $(A \cap B)^c = A^c \cup B^c$ <-- De Morgan's laws

Maybe look at infinite unions and intersections, like on the reals, primarily to emphasize the big-cup and big-cap notation.